FURTHER MATHEMATICS

Paper 9231/11
Further Pure Mathematics

Key messages

Candidates should show all the steps in their solutions, particularly when proving a given result.

Candidates should read questions carefully so that they answer all aspects in adequate depth. They should take note of where exact answers are required.

Candidates should ensure that any sketch graphs are fully labelled and carefully drawn to show behaviour at significant points and limits.

When dividing by a factor, candidates should consider the possibility that the factor may be zero.

Algebraic expressions can often be simplified by the use of common factors.

General comments

The majority of candidates demonstrated good knowledge across the whole syllabus. They showed their working clearly and were accurate in their handling of algebra and calculus. They also showed understanding of transformations.

Comments on specific questions

Question 1

Most candidates showed good knowledge of the structure of an induction proof, though some did not communicate all the steps clearly. Sometimes the proposition was assumed for every integer rather than for $n = k$.

The inductive step was proved either by manipulating $2^{4(k+1)} + 31^{(k+1)} - 2$ or by considering the difference $(2^{4(k+1)} + 31^{(k+1)} - 2) - (2^{4k} + 31^{k} - 2)$ and showing that it was divisible by 15. More complete responses went on to use the assumption to show that $2^{4(k+1)} + 31^{(k+1)} - 2$ must then be divisible by 15.

Some candidates stopped when the algebra was complete, but most knew how to write down the final conclusion.

Question 2

(a) There were many good solutions. Candidates selected the correct formulae and remembered that $\sum_{r=1}^{n} 1 = n$. Like terms were collected correctly. A few did not put in brackets when expanding the sums which led to sign errors.

(b) The most efficient route seen was to start with the fractions on the right and combine them to give the expression on the left. This was almost always correctly done with sufficient detail shown. Some candidates attempted to break up the given expression into partial fractions. If correctly done this took considerable effort. Both proposed fractions should have had Ar + B as the form of the numerator, but often the constant term in the second fraction was assumed to be 0.

The method of differences is clearly well-understood and most candidates wrote three complete terms, including the first and last, so that the pattern of cancellation was clear. Candidates are advised to check that they have written the answer in terms of the correct variable – in this case *n.*

(c) The limit usually followed correctly.

Question 3

(a) There were two methods of approach. Either substituting $x = y^{\frac{1}{3}}$ at the start, rearranging to isolate the radical term and then cubing, or writing $x^4 = 2x^3 + 1$, cubing and then replacing x^3 by y. The substitution needed was well known and any errors were usually in signs. Most knew to pick out the appropriate coefficient to give the sum of the cubes.

(b) Most candidates knew which coefficients to pick out from their new equation to give the sum of the reciprocals. Another method used was to write $S_3 - 8S_2 - 12S_1 - 6S_0 - S_{-1} = 0$ but errors were often made in finding the values of the sums.

(c) The most efficient solutions used the original equation to give $\sum\alpha^4 = 2\sum\alpha^3 + 4$ and their answer for **part (a)**. Some candidates made the substitution $z = x^2$ and found the sum of squares of roots of the new equation.

A common error was to use the wrong equation for the calculations.

Question 4

(a) A pleasing number of candidates could write down the matrix of the rotation in terms of sin60° and cos60°. The matrix of the one-way stretch proved more difficult – the *d* often appeared in the wrong place in the matrix suggesting that there may have been confusion between the terms stretch and shear. A common error was to apply the transformations in the wrong order.

(b) Many knew that the value of the determinant is the multiplying factor for area. The equation $d = \frac{1}{2}d^2$ was not always written down.

(c) Better responses involved pre-multiplying the given **N** by the inverse of their matrix **M**. The method for writing down the inverse of a 2 by 2 matrix seemed well-known, the most common error being in division by the determinant. A few candidates set up and solved simultaneous equations for the elements of the matrix. This was a longer method and often lead to error.

(d) Many candidates used clear notation to distinguish between the object point and its image after transformation by the matrix. The question asked for invariant lines through the origin and most realised that the equation is of the form $y = mx$. A common problem was to cancel the factor $(x + mx)$ when simplifying the equation, losing the solution $m = -1$ and the invariant line $y = -x$.

Question 5

(a) The majority of candidates started the curve at the correct point and drew an arc of approximately the correct shape. A few did not take note of the range of values given for θ and some included a sketch in which representation of the angle $\frac{\pi}{6}$ was disproportionately large. It was rare to see the connection $a\cot\frac{\pi}{6} = 2\sqrt{3}$ used to justify the given answer of $a = 2$.

(b) Most knew the formula for the area, substituted $a = 2$ and used appropriate limits. Integration of $\cot^2\left(\frac{1}{3}\pi - \theta\right)$ proved challenging for many candidates. Those who recognised the relationship $\cot^2\alpha = \operatorname{cosec}^2\alpha - 1$ and that $\operatorname{cosec}^2\alpha$ can be integrated to give $\cot\alpha$ could proceed very efficiently to the final answer. More often substitution and other trigonometric identities were used. This could be a long process until a form was reached that could be integrated, and many became lost in the detail. Candidates should be particularly careful with limits whenever they perform a substitution.

(c) The most elegant solution was to write

$$\cot\left(\frac{1}{3}\pi-\theta\right)=\frac{1}{\tan\left(\frac{1}{3}\pi-\theta\right)}=\frac{1+\tan\frac{1}{3}\pi\tan\theta}{\tan\frac{1}{3}\pi-\tan\theta}$$

Responses that took this approach lead to efficient solutions.

Expanding $\cos\left(\frac{1}{3}\pi-\theta\right)$ and $\sin\left(\frac{1}{3}\pi-\theta\right)$ was usually correct. Sometimes necessary steps were omitted from the manipulation. Whenever the answer is given it is important to show full working and in this case it was helpful to show, for example, that $x=r\cos\theta$ and $y=r\sin\theta$ were being used.

Question 6

In general candidates seem comfortable with the work on vectors and know which formula to use.

(a) Those who used the formula for the distance of a point from a plane were largely successful, although there were some sign errors in vector products.

Another method was to use a general point *P* on l_1, a point *Q* on l_2, and the fact that the dot products of the vector *PQ* with the two line directions is zero, to generate linear equations. These could be combined with the given length of *PQ* to find *t*. This gave scope for introducing errors in algebraic manipulation but showed understanding of the situation.

(b) Many candidates seemed unfamiliar with the parametric vector form of the equation of a plane which was asked for in the question.

(c) Most candidates knew how to find the angle between the line and the normal to the plane and did so correctly. Many candidates did not then go on to find the angle between the line and the plane, omitting the final step involving subtraction from 90°.

(d) This part of the question was often completed accurately.

Question 7

(a) This part was well done. The method for finding the oblique asymptote is well understood and the asymptotes were written as equations.

(b) Those candidates who differentiated the rearranged form used in finding the oblique asymptote did so accurately. Those who used the original form and quotient rule were usually correct, although they should remember that the denominator is part of the expression for the derivative. A few made numerical errors in finding the stationary points and some did not go on to find the *y* coordinates.

(c) Those who drew and labelled axes and asymptotes carefully scored well on this part. Some of the curve sketching needed more precision, particularly in the approach to the asymptotes. The intersection with the *y* axis (0, 9) was rarely forgotten.

(d) The idea of reflecting the previous graph in the *x*-axis was clearly well understood. For the graph to be properly drawn the reflected asymptote also needed to be shown. Those candidates who worked with equations to find the critical points usually had fewer problems in giving the final inequalities. A well-drawn graph helped to interpret the situation correctly and could be referred to in explaining why one branch gave no acceptable solutions for the inequality, instead of considering the algebra.

FURTHER MATHEMATICS

Paper 9231/12
Further Pure Mathematics

Key messages

Candidates should show all the steps in their solutions, particularly when proving a given result.

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When dividing by a factor, candidates should consider the possibility that the factor may be zero.

Algebraic expressions can often be simplified by the use of common factors.

General comments

The majority of candidates demonstrated good knowledge across the whole syllabus. They showed their working clearly and were accurate in their handling of algebra and calculus. They also showed understanding of transformations.

Comments on specific questions

Question 1

Most candidates showed good knowledge of the structure of an induction proof, though some did not communicate all the steps clearly. Sometimes the proposition was assumed for every integer rather than for $n = k$.

The inductive step was proved either by manipulating $2^{4(k+1)} + 31^{(k+1)} - 2$ or by considering the difference $(2^{4(k+1)} + 31^{(k+1)} - 2) - (2^{4k} + 31^{k} - 2)$ and showing that it was divisible by 15. More complete responses went on to use the assumption to show that $2^{4(k+1)} + 31^{(k+1)} - 2$ must then be divisible by 15.

Some candidates stopped when the algebra was complete, but most knew how to write down the final conclusion.

Question 2

(a) There were many good solutions. Candidates selected the correct formulae and remembered that $\sum_{r=1}^{n} 1 = n$. Like terms were collected correctly. A few did not put in brackets when expanding the sums which led to sign errors.

(b) The most efficient route seen was to start with the fractions on the right and combine them to give the expression on the left. This was almost always correctly done with sufficient detail shown. Some candidates attempted to break up the given expression into partial fractions. If correctly done this took considerable effort. Both proposed fractions should have had A*r* + B as the form of the numerator, but often the constant term in the second fraction was assumed to be 0.

The method of differences is clearly well-understood and most candidates wrote three complete terms, including the first and last, so that the pattern of cancellation was clear. Candidates are advised to check that they have written the answer in terms of the correct variable – in this case *n.*

(c) The limit usually followed correctly.

Question 3

(a) There were two methods of approach. Either substituting $x = y^{\frac{1}{3}}$ at the start, rearranging to isolate the radical term and then cubing, or writing $x^4 = 2x^3 + 1$, cubing and then replacing x^3 by *y*. The substitution needed was well known and any errors were usually in signs. Most knew to pick out the appropriate coefficient to give the sum of the cubes.

(b) Most candidates knew which coefficients to pick out from their new equation to give the sum of the reciprocals. Another method used was to write $S_3 - 8S_2 - 12S_1 - 6S_0 - S_{-1} = 0$ but errors were often made in finding the values of the sums.

(c) The most efficient solutions used the original equation to give $\sum\alpha^4 = 2\sum\alpha^3 + 4$ and their answer for **part (a)**. Some candidates made the substitution $z = x^2$ and found the sum of squares of roots of the new equation.

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(a) A pleasing number of candidates could write down the matrix of the rotation in terms of sin60° and cos60°. The matrix of the one-way stretch proved more difficult – the *d* often appeared in the wrong place in the matrix suggesting that there may have been confusion between the terms stretch and shear. A common error was to apply the transformations in the wrong order.

(b) Many knew that the value of the determinant is the multiplying factor for area. The equation $d = \frac{1}{2}d^2$ was not always written down.

(c) Better responses involved pre-multiplying the given **N** by the inverse of their matrix **M**. The method for writing down the inverse of a 2 by 2 matrix seemed well-known, the most common error being in division by the determinant. A few candidates set up and solved simultaneous equations for the elements of the matrix. This was a longer method and often lead to error.

(d) Many candidates used clear notation to distinguish between the object point and its image after transformation by the matrix. The question asked for invariant lines through the origin and most realised that the equation is of the form *y* = *mx*. A common problem was to cancel the factor (*x* +*mx*) when simplifying the equation, losing the solution *m* = –1 and the invariant line *y* = –*x*.

Question 5

(a) The majority of candidates started the curve at the correct point and drew an arc of approximately the correct shape. A few did not take note of the range of values given for *θ* and some included a sketch in which representation of the angle $\frac{\pi}{6}$ was disproportionately large. It was rare to see the connection $a\cot\frac{\pi}{6} = 2\sqrt{3}$ used to justify the given answer of *a* = 2.

(b) Most knew the formula for the area, substituted *a* = 2 and used appropriate limits. Integration of $\cot^2\left(\frac{1}{3}\pi - \theta\right)$ proved challenging for many candidates. Those who recognised the relationship $\cot^2\alpha = \operatorname{cosec}^2\alpha - 1$ and that $\operatorname{cosec}^2\alpha$ can be integrated to give $\cot\alpha$ could proceed very efficiently to the final answer. More often substitution and other trigonometric identities were used. This could be a long process until a form was reached that could be integrated, and many became lost in the detail. Candidates should be particularly careful with limits whenever they perform a substitution.

(c) The most elegant solution was to write

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Responses that took this approach lead to efficient solutions.

Expanding $\cos\left(\frac{1}{3}\pi-\theta\right)$ and $\sin\left(\frac{1}{3}\pi-\theta\right)$ was usually correct. Sometimes necessary steps were omitted from the manipulation. Whenever the answer is given it is important to show full working and in this case it was helpful to show, for example, that $x = r\cos\theta$ and $y = r\sin\theta$ were being used.

Question 6

In general candidates seem comfortable with the work on vectors and know which formula to use.

(a) Those who used the formula for the distance of a point from a plane were largely successful, although there were some sign errors in vector products.

Another method was to use a general point *P* on l_1, a point *Q* on l_2, and the fact that the dot products of the vector *PQ* with the two line directions is zero, to generate linear equations. These could be combined with the given length of *PQ* to find *t*. This gave scope for introducing errors in algebraic manipulation but showed understanding of the situation.

(b) Many candidates seemed unfamiliar with the parametric vector form of the equation of a plane which was asked for in the question.

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(d) This part of the question was often completed accurately.

Question 7

(a) This part was well done. The method for finding the oblique asymptote is well understood and the asymptotes were written as equations.

(b) Those candidates who differentiated the rearranged form used in finding the oblique asymptote did so accurately. Those who used the original form and quotient rule were usually correct, although they should remember that the denominator is part of the expression for the derivative. A few made numerical errors in finding the stationary points and some did not go on to find the *y* coordinates.

(c) Those who drew and labelled axes and asymptotes carefully scored well on this part. Some of the curve sketching needed more precision, particularly in the approach to the asymptotes. The intersection with the *y* axis (0, 9) was rarely forgotten.

(d) The idea of reflecting the previous graph in the *x*-axis was clearly well understood. For the graph to be properly drawn the reflected asymptote also needed to be shown. Those candidates who worked with equations to find the critical points usually had fewer problems in giving the final inequalities. A well-drawn graph helped to interpret the situation correctly and could be referred to in explaining why one branch gave no acceptable solutions for the inequality, instead of considering the algebra.

FURTHER MATHEMATICS

Paper 9231/13
Further Pure Mathematics

Key messages

Candidates should show all the steps in their solutions, particularly when proving a given result.

Candidates should read questions carefully so that they answer all aspects in adequate depth. They should take note of where exact answers are required.

Candidates should ensure that any sketch graphs are fully labelled and carefully drawn to show significant points and behaviour at limits.

When dividing by a factor, candidates should consider the possibility that the factor may be zero.

Algebraic expressions can often be simplified by the use of common factors.

General comments

The majority of candidates demonstrated good knowledge across the whole syllabus. They showed their working clearly and were accurate in their handling of algebra and calculus. They also showed understanding of transformations.

Comments on specific questions

Question 1

(a) Some candidates adopted the elegant approach detailed in the mark scheme, but most chose a more complicated route, often deriving the result without error. Expanding $\tan(r + 1)$ was a popular start but needed several stages and care with multiple fractions to get to the result.

(b) Most recognised the connection with **part (a)** and obtained the correct result. Candidates needed to show three complete terms including first and last. Some candidates worked with r instead of n or misplaced or omitted sin1. A few candidates attempted a solution based on partial fractions.

(c) Better responses recognised that the oscillation of $\tan(n + 1)$ prevented convergence. Many presumed that the terms or the sum tended towards infinity.

Question 2

(a) This was well answered. Only a few did not recognise that the coefficient of x is zero when applying the formula for sum of squares.

(b) (i) Many candidates could write down an expression connecting S_{n+3}, S_{n+2} and S_n. The question asked for this to be expressed in the form $S_{n+3} =$, and this was not done by all candidates.

(ii) Most used their relationship from the previous part to find the value of S_4. Those who found S_3 first were slightly less likely to make errors than those who combined two uses of the formula. A small number of candidates performed another substitution to find an equation whose roots are the squares, and worked from there.

(c) The correct substitution was generally chosen, and the majority produced the right equation by accurate algebra.

(d) The most popular method adopted by candidates was to pick out the coefficients from the answer for **part (c)** and this was almost always correctly done.

Question 3

(a) Most candidates showed good knowledge of the structure of an induction proof, though some did not communicate all the steps clearly. Candidates should write down the algebraic statement of the proposition; sometimes it was assumed for every integer rather than for $n = k$.

The most straightforward method was to add the $(k + 1)$ term and simplify the new sum. Taking out common factors helped in the progress towards the required form of the sum to $(k + 1)$ terms. Those who multiplied out all brackets needed to compare their expression with the sum given by the hypothesis. Some candidates stopped when the algebra was complete, but most knew how to write down the final conclusion.

(b) Almost all picked out the correct formula for $\sum r^2$. The question asked for the answer to be fully factorised and the most straightforward way to achieve this was to take out as many common factors as possible. The answer could then be expressed as a product of linear factors and one quadratic factor, all with integer coefficients, and with the product of factors over a single common denominator or multiplied by a single fraction. Those who decided to multiply out all the brackets, collect terms and then re-factorise rarely managed to complete the task. A common error was to leave brackets within brackets.

Question 4

(a) This part was very well done with only a few slips in arithmetic.

(b) This was generally straightforward. A significant minority found the determinant of **CAB** instead of **A**.

(c) Many candidates used clear notation to distinguish between the object point and its image after transformation by the matrix. The question asked for invariant lines through the origin and most realised that the equation is of the form $y = mx$. A small number found the invariant lines by first finding the eigenvectors of the matrix.

Question 5

(a) Most candidates sketched a correctly shaped curve passing through the pole with the correct domain. The best responses showed clearly that the curve was approximately tangential to the initial line at O.

(b) There were many near perfect solutions to this question. Some candidates combined the terms into a single fraction (a few recovered using substitution) and then tried integration by parts. A few were unable to integrate $\dfrac{1}{(\pi-\theta)^2}$ correctly after the first expansion, or made one or more sign or coefficient errors when integrating. A significant minority of candidates did the integration correctly and then gave insufficient detail in the substitution of the limits and in the combination of the two natural log terms. As always, when candidates are asked to 'show that' an expression (in this case for the area of a region) is correct, complete working must be shown.

Question 6

In vector questions candidates need to check their work carefully as errors in arithmetic can cause problems as the work progresses. Candidates should make sure they distinguish between the position vector of a point and the direction vector of a line.

(a) Many candidates seemed unfamiliar with the parametric vector form of the equation of a plane which was asked for in the question. Several gave the Cartesian equation.

(b) This was generally well done with only a few errors in arithmetic

(c) Most knew what to do here but errors in arithmetic and sign errors in vector products meant that many did not gain accuracy marks.

(d) This was a demanding question for many candidates. Those who adopted the expected approach generally did very well. Some candidates used the vector equation of the plane and two dot products. This was sometimes successful, but a more complex approach led to the greater chance of error. A few made the response more complicated by involving an extra point, often then going on to find the foot of the wrong perpendicular. When introducing an extra point, candidates should define this as, for example, a general point of the plane. Those who worked with the distance of a point from a plane rarely connected this with the requirements of the question to achieve a complete solution.

Question 7

(a) This was generally well done with the exact equations of the asymptotes written correctly.

(b) The differentiation to find the stationary points was performed correctly and the x coordinate found. Occasionally the y coordinate was omitted.

(c) Those who drew and labelled axes and asymptotes carefully scored well on this part. Some of the curve sketching needed more precision, particularly in the approach to the asymptotes. The intersections with axes were usually correct.

(d) The idea of reflecting the graph from **part (c)** was clearly well understood. For the graph to be properly drawn the reflected asymptote also needed to be shown. Those candidates who worked with equations to find the critical points usually had fewer problems in giving the final inequalities.

FURTHER MATHEMATICS

Paper 9231/21
Further Pure Mathematics

Key messages

- Candidates should show all the steps in their solutions, particularly when proving a given result.
- Candidates should read questions carefully so that they answer all aspects in adequate depth, particularly when a geometric interpretation is required.
- Candidates should make use of results derived or given in earlier parts of a question.

General comments

Most candidates demonstrated very good knowledge across the whole syllabus. They showed their working clearly and were accurate in their handling of algebra and calculus. They also showed understanding of linear algebra. Sometimes candidates did not fully justify their answers and jumped to conclusions without justification, particularly where answers were given within the question. Gaps in knowledge were evident in some scripts.

Comments on specific questions

Question 1

(a) The majority of candidates calculated the determinant accurately to show that the system of equations has a unique solution. A few candidates accepted an incorrect determinant without checking their work. Better responses gave a complete and correct geometric description.

(b) Almost all candidates substituted the values into the first equation given in **part (a),** which led to the correct value of *a*.

Question 2

(a) Almost all knew how to approach this question and completed it to a high standard. There was some inaccuracy when solving linear equations to find the particular integral and some problems with notation. A few candidates gave expressions instead of equations as their answer.

(b) Most candidates successfully stated an approximate solution using their answer to **part (a)**.

Question 3

(a) Most candidates formed a correct expression for the sum of the areas of the rectangles and applied the standard result for the sum of cubes to accurately derive the given result.

(b) Most candidates correctly adapted their solution to **part (a)** and derived a suitable lower bound.

(c) Better responses gave a complete argument, fully simplifying $U_n - L_n$ to derive $n > 1000$ and stating 1001 as the least value of *n*.

Question 4

The majority of candidates divided both sides of the equation by $\sin\theta$ and found the integrating factor correctly. After multiplying both sides of the equation by $\tan\frac{1}{2}\theta$, some candidates were able to apply a suitable identity in order to integrate the right hand side and maintained accuracy when substituting in the initial conditions.

Question 5

(a) Almost all could state the sum of the series in terms of z and n.

(b) Better responses stated that $z^n = 1$ and clearly derived the given result from the sum of the series in **part (a)**.

(c) Almost all knew that de Moivre's theorem related the series to the geometric progression in **part (a)**. However**,** it was common to see extra variables, such m or n, in the sum to infinity of the geometric progression which overcomplicated the expression and hindered progress. More successful responses involved finding the real part of $\frac{z}{1-z}$ which, after simplifying the numerator and denominator, led to the given answer.

Question 6

(a) This part of the question was well done, though a few candidates accepted zero eigenvectors without checking for errors in their working. A few candidates spent time finding $\det(\mathbf{A} - \lambda\mathbf{I})$ instead of reading directly from the diagonal of the matrix.

(b) The better responses maintained accuracy throughout their solution, both when manipulating the characteristic equation and when substituting in for $\mathbf{A}$.

Question 7

(a) The majority of candidates accurately differentiated both sides of the equation implicitly and showed enough working to justify the given answer.

(b) Good candidates accurately used implicit differentiation again to find an equation involving the second derivative. For candidates who spotted that $\mathrm{f}(0) = \cosh^{-1} 2$, finding the logarithmic did not pose any difficulties. Better responses showed this connection and also maintained accuracy when substituting to find $\mathrm{f}'(0)$ and $\mathrm{f}''(0)$.

Question 8

(a) Almost all candidates correctly wrote down the derivative of x in terms of t and clearly applied the double angle formula to justify the given derivative of y.

(b) (i) Most candidates accurately recalled the formula for surface area with correct limits. Some candidates were able to fully simplify $\sqrt{\dot{x}^2 + \dot{y}^2}$ before substituting into the formula, which caused fewer errors and enabled a clear path to the given answer.

(ii) After multiplying out $\left(\frac{3}{2}t - \frac{1}{4}\sinh 2t\right)(1 + \cosh 2t)$, the better responses maintained accuracy when applying integration by parts and the necessary hyperbolic identities. Many forms of the correct answer were seen due to the different forms of the double angle formulae required. A few candidates gave an answer involving $\cosh 4$, neglecting to convert their answer to one in terms of $\sinh 2$ and $\cosh 2$ as requested in the question.

FURTHER MATHEMATICS

Paper 9231/22
Further Pure Mathematics

Key messages

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(a) Almost all could state the sum of the series in terms of z and n.

(b) Better responses stated that $z^n = 1$ and clearly derived the given result from the sum of the series in **part (a)**.

(c) Almost all knew that de Moivre's theorem related the series to the geometric progression in **part (a)**. However**,** it was common to see extra variables, such m or n, in the sum to infinity of the geometric progression which overcomplicated the expression and hindered progress. More successful responses involved finding the real part of $\frac{z}{1-z}$ which, after simplifying the numerator and denominator, led to the given answer.

Question 6

(a) This part of the question was well done, though a few candidates accepted zero eigenvectors without checking for errors in their working. A few candidates spent time finding $\det(\mathbf{A}-\lambda\mathbf{I})$ instead of reading directly from the diagonal of the matrix.

(b) The better responses maintained accuracy throughout their solution, both when manipulating the characteristic equation and when substituting in for $\mathbf{A}$.

Question 7

(a) The majority of candidates accurately differentiated both sides of the equation implicitly and showed enough working to justify the given answer.

(b) Good candidates accurately used implicit differentiation again to find an equation involving the second derivative. For candidates who spotted that $\mathrm{f}(0) = \cosh^{-1} 2$, finding the logarithmic did not pose any difficulties. Better responses showed this connection and also maintained accuracy when substituting to find $\mathrm{f}'(0)$ and $\mathrm{f}''(0)$.

Question 8

(a) Almost all candidates correctly wrote down the derivative of x in terms of t and clearly applied the double angle formula to justify the given derivative of y.

(b) (i) Most candidates accurately recalled the formula for surface area with correct limits. Some candidates were able to fully simplify $\sqrt{\dot{x}^2 + \dot{y}^2}$ before substituting into the formula, which caused fewer errors and enabled a clear path to the given answer.

(ii) After multiplying out $\left(\frac{3}{2}t - \frac{1}{4}\sinh 2t\right)(1+\cosh 2t)$, the better responses maintained accuracy when applying integration by parts and the necessary hyperbolic identities. Many forms of the correct answer were seen due to the different forms of the double angle formulae required. A few candidates gave an answer involving $\cosh 4$, neglecting to convert their answer to one in terms of $\sinh 2$ and $\cosh 2$ as requested in the question.

FURTHER MATHEMATICS

Paper 9231/23
Further Pure Mathematics

Key messages

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- Candidates should read questions carefully so that they answer all aspects in adequate depth and note which algebraic form answers are required to take.
- Candidates should make use of results derived or given in earlier parts of a question.

General comments

The majority of candidates demonstrated very good knowledge across the whole syllabus. They showed their working clearly and were accurate in their handling of algebra and calculus. They also showed understanding of linear algebra. Sometimes candidates did not fully justify their answers and jumped to conclusions without justification, particularly where answers were given within the question. There were many scripts of a very high standard.

Comments on specific questions

Question 1

(a) Almost all candidates stated the correct values of *a* and *b*.

(b) Most candidates found the fifth roots of 1 and the cube roots of i. Some candidates reversed the values of *a* and *b*. Better responses showed clear working, starting from 1 and i in exponential form, and listed all eight roots in the required form.

Question 2

The most common approach was to find expressions for the first four derivatives and then substitute $x = 0$ into the derivatives. Better responses maintained accuracy when differentiating and substituting. A few candidates successfully substituted the Maclaurin's series for $\cosh x - 1$ into the Maclaurin's series for $\ln(1 + x)$.

Question 3

(a) Almost all candidates integrated correctly to find an expression for the area under the curve. Better responses formed a correct expression for the sum of the areas of the rectangles, clearly stating the height of the first rectangle, and fully justified the given answer.

(b) Most candidates adapted their solution to **(a)** and derived a suitable lower bound. A few candidates successfully considered the translation $y = x + 1$ and integrated between 0 and N.

Question 4

After expanding $\left(z \pm z^{-1}\right)^5$ using the binomial expansion, better responses grouped terms together clearly before applying the identities $z^n + z^{-n} = 2\cos n\theta$ and $z^n - z^{-n} = 2\mathrm{i}\sin n\theta$ to fully justify their answer. Alternatively, a few candidates applied de Moivre's theorem to each term, and were usually successful.

Question 5

(a) The majority of candidates maintained accuracy when differentiating and substituting to determine the correct value of k.

(b) Almost all knew how to approach this question and completed it to a high standard. There was some inaccuracy when solving linear equations to find the values of the constants and some problems with notation. A few candidates gave expressions instead of equations as their answer.

Question 6

(a) Better responses worked from the left to right when proving the given result, showing full working when expanding $2\sinh^2 x$ in terms of exponentials.

(b) Most candidates found the integrating factor correctly and, after multiplying both sides of the equation by $\sinh x$, were able to apply the result given in **(a)**. Better responses maintained accuracy throughout, particularly when substituting in the initial conditions.

Question 7

(a) Almost all recognised the integral as $\sinh^{-1}\frac{x}{2}$ and, after substituting in the limits, fully simplified the logarithmic form to $\ln 2$.

(b) The majority of candidates accurately differentiated the product given in the question. Those who then replaced x^2 with x^2+4-4 successfully derived the given reduction formula. Others who integrated $x^2\left(4+x^2\right)^{-\frac{1}{2}n-1}$ by parts were less successful.

(c) This part was well done with the majority of candidates accurately applying the reduction formula.

Question 8

(a) This part was answered well by most candidates. A few candidates incorrectly stated $a \neq \frac{8}{3}$.

(b) This part was answered well by most candidates.

(c) This part of the question was well done, though a few candidates accepted zero eigenvectors without checking for errors in their working. Better responses showed full working when factorising $\det(\mathbf{A}-\lambda\mathbf{I})$.

(d) Better responses maintained accuracy and use correct notation when manipulating the characteristic equation. Some candidates spent time substituting in for **A** and $\mathbf{A}^2$, which was not required to answer the question.

FURTHER MATHEMATICS

Paper 9231/31
Further Mechanics

Key messages

A diagram is often a useful tool in helping candidates to make good progress. This is particularly the case when forces or velocities are involved. If a diagram is given on the question paper, then it may be sufficient to annotate that diagram, but candidates are always free to draw their own diagram as well.

When a result is given in a question, candidates must take care to provide sufficient detail in their working, so that there is no doubt that the offered solution is clear and complete. However, in all questions candidates are advised to show all steps in their working, as credit is given for method as well as accuracy.

General comments

In most questions, the majority of candidates understood what method to use. Those candidates who drew a suitable diagram, annotated appropriately, generally went on to form correct equations. For example in **Question 2**, candidates who drew a clear diagram labelled with all forces and distances used it effectively to correctly set up Newton's equation for horizontal and vertical motion.

Candidates are strongly encouraged to define the symbols they are using, unless specified in the text.

Candidates are reminded that, when the answer is given, they are expected to show their working in full, even if it involves the use of simple algebra.

Comments on specific questions

Question 1

Many candidates were able to set up a differential equation, successfully integrate it and obtain the correct answer after determining the value of the constant of integration. Many examples of good integration skills were seen.

The most common mistake was to omit the negative sign in the differential equation, leading to the determination of an incorrect value for the constant of integration and, therefore, an incorrect solution.

Some candidates did not express the acceleration as derivative of the velocity, some did not separate the variables, and some did not correctly evaluate $\int \frac{1}{(t+1)^2}dt$, a common mistake being $\frac{1}{(t+1)^2}dt = \ln(t+1)^2$.

Question 2

This question proved challenging, and few candidates wrote the two equations for the equilibrium of forces, going on to express the relationship between the radius of the circle and that of the bowl. Many candidates who wrote the three equations correctly managed to use them to obtain a quadratic equation, in terms of either the unknown height x or the angle θ between the reaction force and the vertical (or the horizontal). Some solutions were very elegant and led to the correct answer in a few steps, showing good algebraic manipulative skills. One such approach combined the three equations into $\frac{8}{3} = \frac{(\sin\theta)^2}{\cos\theta}$, used the

Pythagorean identity to obtain and solve the equation $3(\cos\theta)^2 + 8\cos\theta - 3 = 0$ and finally obtained the correct value for the height x.

Question 3

(a) Most candidates correctly applied and rearranged Hooke's Law. Some candidates applied Hooke's Law, but then did not show the steps required to obtain the given answer. Candidates should be reminded that when the answer is given, they should put particular care into showing their working.

(b) Most candidates identified the need to use conservation of energy and wrote the corresponding equation with various degrees of success. Most of them obtained correct expressions for the kinetic energy (KE) and gravitational potential energy (GPE). However, the correct identification of the two terms of the elastic potential energy (EPE) proved challenging. Stronger responses simplified part of the expression for the EPE; $\left(x - \frac{2a}{3}\right)^2 - (x-a)^2 = \frac{2}{3}ax - \frac{5}{9}a^2$. Many candidates demonstrated skilful manipulation of the energy terms to obtain a linear equation in x and a leading, almost always, to the correct answer. Some candidates used the result from **part (b)** to simplify one of the terms of EPE obtaining the equation $\frac{1}{3}mga + \frac{2mg}{x-a}\left(x - \frac{2a}{3}\right)^2 - 2mg(x-a) = 2mga$ but did not multiply all terms by $(x-a)$ to eliminate the denominator. Another common error was to use m instead of $6m$ for the expressions of KE and GPE.

Question 4

(a) This part was answered correctly by many candidates. They correctly calculated the moments and confidently manipulated the equation to obtain the correct answer. A typical error was to express the distance of the centre of mass of the cone from the base of the cylinder as $\left(\frac{3}{4}kh + h\right)$ instead of $\left(\frac{1}{4}kh + h\right)$.

(b) Stronger responses contained a correct sketch of a diagram to answer this part. With the aid of the diagram, candidates were able to obtain the equation linking θ, r and the distance obtained in the previous part. They then confidently solved the equation and excluded the negative value for k. Candidates who did not draw a diagram were rarely successful on this part.

Question 5

To obtain the correct answer for this part, candidates were required to perform three steps: to combine Newton's Second Law at points A and B using the given relationship between the tensions at the two points; to find a first equation in u,a, g, and θ and to use the conservation of energy to derive a second equation in the same variables; and finally to combine the two equations to obtain the answers. The first step proved more challenging than the second one for many candidates, with typical mistakes involving positive/negative signs in the equations. Many candidates managed to perform the second step correctly. The strongest answers showed excellent algebraic manipulative skills as well as strategic thinking, as these candidates rearranged the two equations as $u^2 = ag(7 - 8\cos\theta)$ and $u^2 = ag(4\cos\theta + 1)$, thus greatly simplifying the third step. A few candidates correctly calculated the value for $\cos\theta$, but then did not work out the value of θ, or did not obtain an expression for u in terms of a and g, after obtaining the correct answer for θ.

Question 6

(a) In answering this question, many candidates showed a great degree of confidence in writing and successfully solving the system of equations obtained from the conservation of linear momentum and Newton's experimental law. They correctly determined that the component of the velocity of sphere A along the line of centres after the collision was zero. However, a few of them could not explain why this implied that the direction of motion of the sphere was perpendicular to the line of centres. Stronger responses were characterised by good algebraic manipulative skills and efficient and effective strategies to solve the system of simultaneous equations.

(b) This part proved more challenging than **part (a)**. Only the minority of candidates could determine the kinetic energy of sphere B after the collision. These candidates usually had no difficulties in solving the equation to obtain the correct answer, at times in a very elegant manner.

Question 7

(a) Many candidates answered this part well. They used a variety of methods, some resulting in very elegant approaches. One of the most efficient methods was to determine the time of the maximum height by solving the equation $100\sin\theta - gt = 0$ ($t = 8$), to then exploit the symmetry of the trajectory and the given interval of 10 seconds to determine the time corresponding to H ($t = 3$ and $t = 13$), and finally to substitute either value into the equation of the height $H = 100\sin\theta - \frac{1}{2}gt^2$ to obtain the correct answer. Some candidates substituted both values into the equation of the height, allowing them to check their answer. This approach is to be encouraged.

(b) This question was answered correctly by many candidates. A typical error was to determine the vertical component of the velocity as 70 (instead of –70) without an accompanying diagram to indicate that the direction was downwards. Some candidates worked out the direction of the velocity as 49.4° and omitted to say that it was below the horizontal or to draw a diagram to that extent. The strongest answers combined together a correct working (including negative signs) with clear accompanying diagrams.

FURTHER MATHEMATICS

Paper 9231/32
Further Mechanics

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(b) Stronger responses contained a correct sketch of a diagram to answer this part. With the aid of the diagram, candidates were able to obtain the equation linking θ, r and the distance obtained in the previous part. They then confidently solved the equation and excluded the negative value for k. Candidates who did not draw a diagram were rarely successful on this part.

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FURTHER MATHEMATICS

Paper 9231/33
Further Mechanics

Key messages

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General comments

In most questions the majority of candidates understood what method to use. Those candidates who drew a suitable diagram, annotated appropriately, generally went on to form correct equations.

Candidates are reminded that, when the answer is given, they are expected to show their working in full, even if it involves the use of simple algebra. This was particularly the case in **Question 6(a)** and **Question 7(c)**.

Comments on specific questions

Question 1

Most candidates answered this question correctly by writing down a moments equation involving the areas of triangles *ABD* and *BCD* and the distance of the centre of mass of each from the line *BD*. Some candidates realised that the ratio of the areas of the triangles was 1:2 and used this instead of calculating the areas. A minority of candidates decided to take moments about the point *A* but then did not subtract $3a$ from their answer.

Question 2

Most candidates showed a clear strategy for answering this question and carried it through successfully. Some candidates made a sign error in the energy equation and a small number used a change in the gravitational potential energy from a point other than the natural length. A few did not identify which value to use for extension in the formula for the elastic potential energy.

Question 3

(a) This part was answered correctly by most candidates.

(b) Most candidates knew that they needed to use Newton's second law for the horizontal component of the motion of particle *A*, but a variety of errors were seen. Some candidates were unsure of the radius of the circle and some wrote that the component of the tension in the string was $T\cos\theta$ instead of $T\sin\theta$. A number of candidates transposed the masses of *A* and *B*.

Question 4

To solve this problem, candidates needed to perform three steps: to write down an energy equation for the motion from *A* to *B*, to apply Newton's second law at *B* and then again at *A*. Most candidates produced these three equations, and many then went on to combine the first two to find either the value of $\tan\theta$ or an expression for u^2 in terms of *a* and *g*. Stronger responses found the tension in the string at *A* by substituting into the third equation. Some candidates worked with all three equations, finding various quantities in terms of others, but with no clear strategy to for obtaining an expression for the required tension in terms of *m* and *g*.

Question 5

(a) Most candidates used Newton's second law to set up a differential equation. They then separated the variables and integrated, evaluated the constant of integration and rearranged to find *v* in terms of *t*. Some candidates had a sign error in the differential equation and some introduced errors in manipulating the logarithms. A minority of candidates used SUVAT equations throughout, but as the force is variable, the acceleration is not constant.

(b) Most candidates integrated their result from **part (a)** to find *x* in terms of *t*. However, some candidates restarted and used $v\frac{\mathrm{d}v}{\mathrm{d}x}$ as the acceleration, finding *v* in terms of *x*, before referring back their answer to **part (a)**. This was a possible route, but was unnecessarily long and complicated and was rarely successful.

(c) This part was completed successfully by many candidates.

Question 6

(a) Most candidates gave correct equations for the conservation of momentum and Newton's experimental law. They then had to solve them to show the result for the speed of *B* after the collision. Most candidates were able to do this, but with varying degrees of complexity. At one extreme, some candidates wrote down the result without any working. This is not sufficient in a 'show that' question. At the other extreme, some candidates filled the page with working, often taking a more complex route than required, or finding the speed of *A* first.

(b) There were some excellent solutions to this part, usually structured and neatly presented. A common error was to omit the component of the velocity of sphere *A* perpendicular to the line of centres. A second common error was to misinterpret the phrase "70 per cent of the total kinetic energy of the spheres is lost" as meaning the final kinetic energy is 70 per cent of the initial kinetic energy. The algebraic manipulation required in isolating the value of *k* often led to errors which may have been avoided if the presentation of work had been clearer.

Question 7

There were some very good solutions to this question, but few candidates followed the method suggested rather than the more familiar SUVAT route for finding expressions for the range and the greatest height of a projectile. Some candidates quoted the results without any supporting working.

(a) Many candidates wrote down the equation of the trajectory and substituted $y = 0$, as suggested. They then went on to take out the common factor in the resulting expression and rearranged. Any correct form of the expression for *R* was acceptable. It was not necessary to continue until the familiar expression was deduced.

(b) Stronger responses used the fact that the greatest height corresponds to half of the range, so that the trajectory equation could be used again together with the result of **part (a)** to reach the required expression.

(c) Candidates who combined the results of **parts (a)** and **(b)** were most successful in reaching an efficient result. Sufficient working was required for this 'show that' question. Many candidates substituted their expressions for *R* and *H* into the given expression followed immediately by $\theta = 60°$.

(d) There were some well thought out solutions and a wide variety of approaches to this part. Some excellent mathematical thinking was seen. Most candidates followed the suggestion to differentiate the equation of the trajectory, but some did not know how to make any further progress. Stronger responses identified that the limiting cases correspond to $\frac{dy}{dx} = \pm 1$, leading to $x = \sqrt{3} \pm 1$ and the inequality $\sqrt{3} - 1 < x < \sqrt{3} + 1$. A range of different approaches were seen for this part, many of which were acceptable.

FURTHER MATHEMATICS

Paper 9231/41
Further Probability and Statistics

Key messages

Candidates are advised to show all their working, as credit is given for method as well as accuracy. When a result is given in a question, candidates must give sufficient detail in their working, so that there is no doubt that the offered solution is clear and complete.

Care must be taken with the language used when interpreting the result of any test. In general, a hypothesis test is not a proof and it is not appropriate to use definitive statements. Concluding statements should always include some degree of uncertainty, for example, "there is insufficient evidence to support the claim that…." rather than "the test proves that….".

General comments

The standard of candidates' work was generally high, with many presenting clear and accurate solutions throughout.

The rubric for this paper specifies that non-exact numerical answers should be given to 3 significant figures. Candidates should work to a greater degree of accuracy in the steps leading to the final answer. Premature rounding to only 3 significant figures may result in an error in the third figure in the final answer. This is particularly the case in statistics problems where several different values are calculated, each depending on the previous one. Such rounding errors were often seen in **Question 2** and **Question 4**.

Comments on specific questions

Question 1

(a) Almost all candidates knew how to proceed with this question. The final mark was awarded for an appropriate statement of the correct conclusion from the result of the hypothesis test. There were some common errors which led to this mark not being awarded. Some candidates did not state a conclusion in words, simply saying 'reject H_0'. Some candidates gave a definitive statement such as "$\mu < 8.22$". In interpreting results, there needs to be a level of uncertainty in the language used, so for example, "there is insufficient evidence to accept the null hypothesis $\mu = 8.22$" or "there is sufficient evidence to accept the alternative hypothesis $\mu < 8.22$".

(b) Many candidates correctly stated that the population or underlying distribution needed to be normal, but a common error was to say that the data or simply that "it" should be normally distributed which was not precise enough. Incorrectly stating that it was the sample or the population mean that should be normal, was also seen.

Question 2

The calculation of the test statistic was generally done accurately. Many candidates did not state the hypotheses in sufficient detail. The null hypothesis needed to be equivalent to "driving test success is independent of instructor". Many candidates gave statements that were lacking in detail, for example, "success is independent" or "the test is independent". Other candidates reversed the null and alternative hypotheses. As in other questions, a level of uncertainty in the final conclusion was often lacking.

Question 3

There were many fully correct solutions to this question.

(a) This was usually answered correctly. A few candidates used an incorrect expression for $E(\sqrt{X})$ often using $F(x)$ rather than $f(x)$ or using x rather than $\sqrt{x}$ in the required integral.

(b) Again this was usually answered correctly. Candidates who had used incorrect expressions in **part (a)** usually continued with similar errors in this part.

(c) The method for obtaining the probability density function for Y was almost always applied correctly.

Question 4

The majority of candidates made a good attempt at answering this question, but there was sometimes a lack of accuracy in carrying out a correct method, usually through premature approximation. Some candidates used an incorrect method to find the combined variance for the two distributions. The tabular value of 1.96 used in the comparison and the outcome "reject H_0" were usually correct, but conclusions often lacked any level of uncertainty.

Question 5

(a) Only a few candidates gave a reason that was sufficiently correct to gain the mark. Most reasons did not include any mention of differences and/or referred to the sample rather than the population.

(b) Most candidates knew the basic method involved in carrying out a Wilcoxon matched-pairs signed-rank test. They were able to calculate the differences and the signed rankings and then find the sum of each of the negative and positive ranks. It is worth noting that a tabular approach to this is a helpful aid to accuracy. The hypotheses were often not stated correctly, in terms of the difference between population medians. A common error was to refer to the difference of means rather than medians, or simply use the notation μ. Another common error was to use hypotheses such as "uniform X = uniform Y". As in all tests, a level of uncertainty is expected in the concluding statement.

Question 6

All parts of this question were answered well by candidates. Some candidates were unable to find the correct probabilities in **part (a)** and **part (b)** but knew the methods to use in **part (c)** and **part (d)**. Very few numerical and algebraic errors were seen.

FURTHER MATHEMATICS

Paper 9231/42
Further Probability and Statistics

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(b) Many candidates correctly stated that the population or underlying distribution needed to be normal, but a common error was to say that the data or simply that “it” should be normally distributed which was not precise enough. Incorrectly stating that it was the sample or the population mean that should be normal, was also seen.

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FURTHER MATHEMATICS

Paper 9231/43
Further Probability and Statistics

Key messages

In all questions candidates are advised to show all their working, as credit is given for method as well as accuracy. When a result is given in a question, candidates must give sufficient detail in their working, so that there is no doubt that the offered solution is clear and complete.

Care must be taken with the language used when interpreting the result of any test. In general, a hypothesis test is not a proof and it is not appropriate to use definitive statements. Concluding statements should always include some degree of uncertainty, for example, “there is insufficient evidence to support the claim that….” rather than “the test proves that….”.

General comments

Almost all candidates attempted all the questions. The standard was generally high, with many candidates presenting clear and accurate solutions throughout.

The rubric for this paper specifies that non-exact numerical answers should be given to 3 significant figures. Candidates should work to a greater degree of accuracy while working towards the final answer. Premature rounding or working to only 3 significant figures may result in an error in the third figure in the final answer. This is particularly the case in statistics problems where several different values are calculated, each depending on the previous one. Such rounding errors were often seen in **Question 3** and **Question 5(b)**.

Comments on specific questions

Question 1

Almost all candidates knew how to proceed with this question. The final mark was awarded for an appropriate statement of the correct conclusion from the result of the hypothesis test. There were some common errors which led to this mark not being awarded. Some candidates did not state a conclusion in words, simply saying “reject H_0”. Some candidates gave a definitive statement such as “the claim is justified”. In interpreting results, there needs to be a level of uncertainty in the language used, so for example, “there is insufficient evidence to reject Farmer B’s claim” or “there is sufficient evidence to accept Farmer B’s claim”.

Question 2

Most candidates knew the basic method involved in carrying out a Wilcoxon matched-pairs signed-rank test. They were able to calculate the differences and the signed rankings and then find the sum of each of the negative and positive ranks. It is worth noting that a tabular approach to this is a helpful aid to accuracy. Solutions were less successful in other details of the test. The hypotheses were not often stated correctly, in terms of the difference between population medians. One common error was to refer to the difference of means rather than medians, or simply use the notation μ. Another common error was to use hypotheses such as “there is no difference between the tasters”. As in all tests, a level of uncertainty is expected in the concluding statement.

Question 3

Many candidates scored full marks on this question. The most common error was a loss of accuracy as the calculation progressed, with premature approximation to 3 significant figures at an early stage. This usually led to an inaccurate final answer and could quite easily have been avoided. A minority of candidates made the implicit assumption that the population variances of the two distributions were equal and used a pooled variance estimate. The information given in the question did not justify this assumption.

Question 4

(a) The majority of candidates realised that they needed to differentiate the given probability generating function twice and then to use known formulae to calculate the values of E(X) and Var(X). Many candidates worked through this process accurately and obtained the correct answers of 4 and 24. However, a significant minority of candidates made errors in their differentiation. Having obtained the correct first derivative as $\frac{4t}{\left(3-2t^2\right)^2}$, a common error was to treat the numerator as a constant, leading to the incorrect second derivative $\frac{32t^2}{\left(3-2t^2\right)^3}$. Correct differentiation required the use of either the product rule or the quotient rule.

(b) The most concise solutions to this part involved using the binomial theorem to expand the expression for the probability generating function in powers of t. So,
$G_X(t) = \left(3-2t^2\right)^{-1} = \frac{1}{3}(1+\frac{2}{3}t^2+\frac{4}{9}t^4+\ldots)$. The value of P($X$ = 4) can then be identified as the coefficient of t^4, namely $\frac{4}{27}$.

A significant number of candidates selected a Maclaurin series expansion and this was usually not a useful method. It involves differentiating $G_X(t)$ four times, with more and more terms appearing and almost always, more errors. In this approach, the value of P(X = 4) is $G''''_X(0)$ divided by 4!, but most candidates pursuing this route omitted the 4!. A significant minority of candidates made no meaningful progress in this part.

Question 5

(a) In a Poisson distribution, the mean and variance are equal to each other, so in this part candidates were required to demonstrate that this was at least approximately true for the given information. Squaring the given standard deviation of 1.5 gives 2.25 as the variance and this is sufficiently close to the given mean of 2.4 to suggest that a Poisson distribution may be appropriate. A successful answer to this part required both sight of 2.25 and a statement that this was close to 2.4. However, many candidates gave only one of these two aspects.

(b) Most candidates showed that they knew the principle of a goodness of fit test, but very few offered a fully correct and accurate solution. The first step was to calculate the expected frequencies corresponding to a Poisson distribution with mean 2.5. A very common error was in dealing with the final column in the given table, corresponding to 7 or more breakages. Some candidates took this as exactly 7, others omitted it altogether, presumably because the observed frequency was zero. Both of these errors led to a total expected frequency of less than 180. Candidates are advised that they should always check that their total expected frequency is the same as the total observed frequency. The correct expected frequency for 7 or more is 2.556 and since this is less than 5, the last two columns must be combined. A significant minority of candidates omitted this step, including some who had the correct expected frequencies.

The errors described, together with errors due to premature approximation, led to a wide range of values for the test statistic. Another common error was to use an incorrect tabular value when applying the test, usually coming from an extra degree of freedom.

Question 6

The majority of candidates scored full marks on this question.

(a) The most common error in this part was the omission of a constant term in the second part of the cumulative distribution function. The result of this is that the two parts of the function do not match at the boundary $x = 1$.

(b) Some candidates were not sure as to whether or not they should include a contribution from the first part of the probability density function. This was often seen in several attempts at this part, followed by a decision as to which attempts to cross out.

Some candidates who had made an error in **part (a)** recovered the situation in this part by using the probability density function rather than their incorrect cumulative distribution function.

(c) The method for obtaining the probability density function for Y was almost always applied correctly.